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SUBCRITICAL AND SUPERCRITICAL SHEAR FLOWS ABOVE AN UNEVEN BOTTOM

V. M. Teshukov[1] **and A. Budlal**[2]

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The concepts of subcritical and supercritical flows are introduced for the long-wave approximation model describing stationary free-boundary rotational flows of an ideal incompressible fluid. Shear flows of a fluid layer above an uneven bottom are analyzed. Exact solutions describing different flow regimes are constructed, and the flow properties are studied as a function of the flow regime. Flows with backward streamlines are considered.

Key words: *long-wave approximation, shear flow, free boundary, subcritical and supercritical regimes, recirculation zone.*

1. Equations of Motion. Plane–parallel motions of an ideal incompressible fluid in a free-boundary layer $\eta(x) < y < h(t,x)$ above an uneven bottom are considered. It is assumed that the characteristic vertical scale of the problem H_0 is much smaller than the typical wavelength L_0, so that the ratio H_0/L_0 is much smaller than unity ($H_0/L_0 \ll 1$). In this case, the long-wave model (the shallow-water model) is applicable [1, 2]):

$$\rho(u_t + uu_x + vu_y) + p_x = 0, \qquad p_y = -\rho g,$$
$$u_x + v_y = 0. \tag{1.1}$$

Here ρ is the constant fluid density, p is the pressure, u and v are the horizontal and vertical velocity components, respectively, g is the acceleration due to gravity, t is time, and x and y are two-dimensional Cartesian coordinates. The last equation of system (1.1) allows one to introduce a stream function ψ that obeys the relations

$$\psi_y = u, \qquad \psi_x = -v.$$

The following boundary conditions are imposed on the specified boundary $y = \eta(x)$ and the free boundary $y = h(t,x)$:

$$y = \eta: \qquad v = u\eta_x,$$
$$y = h: \qquad h_t + uh_x = v, \qquad p = p_0 = \text{const}. \tag{1.2}$$

In the long-wave approximation, the condition of no vorticity reduces to the relation $u_y = 0$. If this condition is satisfied, the formulation of the problem (1.1), (1.2) leads to the classical shallow-water equations describing motions with constant velocity over the depth. The present paper studies rotational flows of general form characterized by the inequality $u_y \neq 0$. They will be called shear flows. Free-boundary shear flows are conveniently described using the Eulerian–Lagrangian coordinates t, x, and λ ($0 < \lambda < \lambda_1$, where $\lambda_1 = \text{const}$), which are introduced as follows (see [3]):

$$y = \Phi(t,x,\lambda), \qquad \Phi_t(t,x,\lambda) + u\Phi_x(t,x,\lambda) = v,$$

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[1]Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; teshukov@hydro.nsc.ru. [2]Laboratory of Mechanics, Lille University of Science and Technology, Villeneuve d'Ascq Cedex 59655, France. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 47, No. 4, pp. 26–38, July–August, 2006. Original article submitted September 7, 2005.

$$\Phi(0,x,\lambda)=\frac{\lambda}{\lambda_1}\,h_0(x)+\frac{\lambda_1-\lambda}{\lambda_1}\,\eta(x). \tag{1.3}$$

With this transformation, the unknown free boundary $y=h$ is described by the straight line $\lambda=\lambda_1$ in the plane of the new variables, and the bottom $y=\eta$ by the straight line $\lambda=0$. In the new coordinates, the equations of motion become

$$u_t+uu_x+g\int\limits_0^{\lambda_1}H_x d\lambda+g\eta_x=0,\qquad H_t+(uH)_x=0. \tag{1.4}$$

Here $H(t,x,\lambda)=\Phi_\lambda(t,x,\lambda)$. The depth of the fluid layer δ is linked to $H(t,x,\lambda)$ by the relation

$$\delta=h-\eta=\int\limits_0^{\lambda_1}H\,d\lambda. \tag{1.5}$$

Once the functions $u(t,x,\lambda)$ and $H(t,x,\lambda)$ are found, function $\Phi(t,x,\lambda)$ is obtained by integration

$$\Phi(t,x,\lambda)=\int\limits_0^{\lambda}H(t,x,\lambda')\,d\lambda'+\eta(x),$$

and function $v(t,x,\lambda)$ is determined from the second relation (1.3).

In the case of back transformation to the variables t, x, and y, the dependence $y=\Phi(t,x,\lambda)$ is inverted. Substitution of $\lambda=\lambda(t,x,y)$ into the representation of the obtained solution of the problem (1.1), (1.2) yields the flow parameters in the initial variables t, x, and y. The back transformation is performed uniquely if the condition $H(t,x,\lambda)\neq 0$ is satisfied.

2. Stationary Shear Flows. We consider stationary flows and choose ψ as the variable λ. Integrating Eqs. (1.4), we obtain the relations

$$u^2/2+gh=F(\psi),\qquad H=1/u. \tag{2.1}$$

Here $F(\psi)$ is an arbitrary function. Below, it is assumed to be specified. The last relation in (2.1) follows from the definition of the function ψ:

$$H=y_\psi=1/\psi_y=1/u.$$

Using integrals of the equations of stationary motion, we express the velocity u and the function H in terms of $F(\psi)$ and the unknown quantity h:

$$u=\pm\sqrt{2(F(\psi)-gh)},\qquad H=\pm 1/\sqrt{2(F(\psi)-gh)}. \tag{2.2}$$

We first consider stationary flows in which the function u does not change sign; in this case, for definiteness, we choose $u>0$ and, accordingly, the plus sign in formulas (2.2). Substituting H into relation (1.5), we obtain the following equation for the free-boundary elevation $h=h(x)$:

$$K(h)=\eta(x),\qquad K(h)=h-\int\limits_0^{\psi_1}\frac{d\psi}{\sqrt{2(F(\psi)-gh)}}. \tag{2.3}$$

The function $K(h)$, which plays the main role in the analysis of stationary flows, is defined for $h\in(-\infty,h_0]$, where $h_0=\min\limits_{\psi}F(\psi)/g$. The first and second derivatives of the function $K(h)$ are easily calculated:

$$K'(h)=1-g\int\limits_0^{\psi_1}\frac{d\psi}{(\sqrt{2(F(\psi)-gh)}\,)^3}=1-g\int\limits_0^{\psi_1}\frac{H\,d\psi}{u^2},$$

$$K''(h)=-3g\int\limits_0^{\psi_1}\frac{d\psi}{(\sqrt{2(F(\psi)-gh)}\,)^5}=-3g\int\limits_0^{\psi_1}\frac{H\,d\psi}{u^4}<0.$$

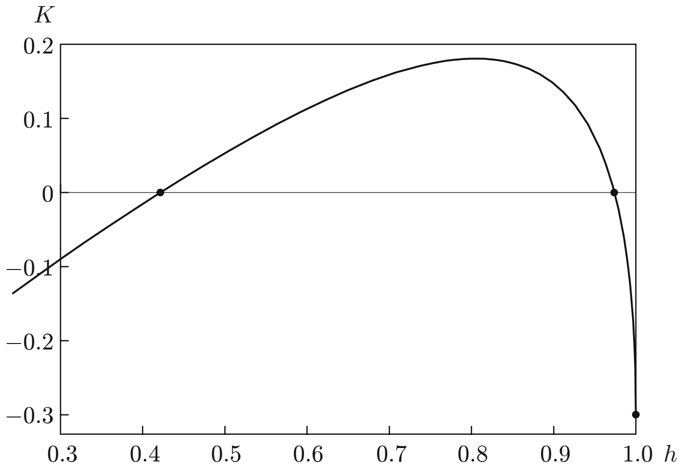

Fig. 1. Dependence $K(h)$.

Let us show that there is a unique value $h = h_c$ $(-\infty < h_c < h_0)$ such that $K'(h_c) = 0$. Indeed, $K'(h)$ changes sign in the semi-infinite interval $(-\infty, h_0]$ because $K'(h) \to 1$ as $h \to -\infty$ and $K'(h) \to -\infty$ as $h \to h_0$. Then, by virtue of the inequality $K''(h) < 0$, there is a unique value $h = h_c$ $(-\infty < h_c < h_0)$ such that $K'(h_c) = 0$ and the function $K(h)$ reaches the maximum value of $K_{\max}$ at the point h_c.

Stationary flows that obey the inequality

$$K'(h) = 1 - g \int\limits_0^{\psi_1} \frac{d\psi}{(\sqrt{2(F(\psi) - gh)}\,)^3} = 1 - g \int\limits_0^{\psi_1} \frac{H\, d\psi}{u^2} < 0 \tag{2.4}$$

will be called subcritical flows and flows that obey the inverse inequality will be called supercritical flows [the equality in (2.4) corresponds to the critical parameters]. We note that in the case of irrotational flows $[F(\psi) = \text{const}]$, the conditions given above coincide with the classical subcritical and supercritical conditions for flows without velocity shear:

$$|u| < \sqrt{g\delta}, \qquad |u| > \sqrt{g\delta}.$$

For the system of equations describing long-wave propagation in rotational flow, Teshukov [2] obtained the following characteristic equation for $k = dx/dt$ — the propagation velocities of generalized characteristics [the wave fronts $x = x(t)$ bounding the regions of perturbed motion]:

$$1 - g \int\limits_0^{\lambda_1} \frac{H\, d\psi}{(u - k)^2} = 0. \tag{2.5}$$

If at a certain x, stationary flow becomes critical, then, according to (2.4) and (2.5), this implies that at the given point, one of the propagation velocities of small perturbations vanishes.

In the interval of $h \in (h_c, h_0)$, which corresponds to subcritical flow, the function $K(h)$ decreases monotonically from $K_{\max}$ to $K_0 = K(h_0)$. In the interval of $h \in (-\infty, h_c)$, which corresponds to supercritical flows, the function $K(h)$ increases monotonically and $-\infty < K(h) < K_{\max}$.

As an example, we consider stationary shear flow with a linear velocity profile over the depth for which

$$F(\psi) = 2.762\psi + 9.81, \qquad \psi_1 = 1.5635. \tag{2.6}$$

The behavior of the function $K(h)$ for this flow is shown in Fig. 1. In the case considered, $h_0 = 1$ and $K(h_0) = -0.3$.

From the properties of the function $K(h)$ listed above, it follows that the equation

$$K(h) = \eta(x) \tag{2.7}$$

is solvable in the range of values of the variable x in which the inequality $\eta(x) \le K_{\max}$ is satisfied. In this case, in the subregion $K_0 \le \eta(x) \le K_{\max}$, Eq. (2.7) has two and only two solutions: $h = h_1(x)$ and $h = h_2(x)$. The first solution corresponds to subcritical flow, and the second to supercritical flow, and $h_2(x) < h_c < h_1(x) \le h_0$.

For x satisfying the inequality $\eta(x) < K_0$, there is only one "supercritical" branch of the solution $h = h(x)$. If the specified function $\eta(x)$ satisfying the inequality $\eta(x) \le K_{\max}$ reaches the maximum value $\eta_{\max} = K_{\max}$ at some points, transcritical flows with a change in the flow regime can exist at the points where the function $\eta(x)$ reaches maximum values.

Differentiating Eq. (2.7), we obtain the relation

$$h'(x) = \eta'(x)/K'(h),$$

from which it follows that for the subcritical flow regime, the function $h(x)$ decreases (increases) with increasing (decreasing) $\eta(x)$, and for the supercritical regime, $h(x)$ increases (decreases) with increasing (decreasing) $\eta(x)$. Moreover, the thickness of the fluid layer $\delta(x) = h(x) - \eta(x)$ changes similarly since the sign of the quantity

$$\delta'(x) = h'(x) - \eta'(x) = \eta'(x)\Big(\frac{1}{K'(h)} - 1\Big) = g\frac{\eta'(x)}{K'(h)} \int\limits_0^{\psi_1} \frac{H\,d\psi}{u^2}$$

coincides with the sign $h'(x)$. In the supercritical flow regime, the behavior of $h(x)$ and $\delta(x) = h(x) - \eta(x)$ changes since the inequality $K'(h) > 0$ is satisfied in this case. In this flow regime, the function $h(x)$ increases (decreases) with increasing (decreasing) $\eta(x)$. In transcritical transition from one regime to the other at the points where the equality $h = h_c$ holds, the quantifies $\eta'(x)$ and $K'(h)$ change sign simultaneously, and, therefore, the functions $h(x)$ and $\delta(x)$ change monotonically in the vicinity of the indicated points.

As a result, it is established that the supercritical branch of the stationary solution exists for $-\infty < \eta(x) < K_{\max}$ and the subcritical branch exists for $K_0 < \eta(x) < K_{\max}$. In the interval of $\eta \in [K_0, K_{\max}]$, both branches are defined. Once the functions $u(x,\psi)$, $H(x,\psi)$, and $h(x)$ are found, it is possible to determine the solution in the plane of the variables (x, y). For this, we calculate

$$y = \Phi(x,\psi) = \int\limits_0^{\psi} H(x,\psi')\,d\psi' + \eta(x) \qquad (2.8)$$

and, inverting the dependence obtained, we find the function $\psi = \psi(x, y)$ and then the function $u = u(x, \psi(x, y))$. The quantity v is calculated similarly. We first determine $v(x,\psi)$ using formulas (2.8) and (1.3), and then $v = v(x, \psi(x, y))$.

3. Flow over a Local Obstacle at a Channel Bottom. We consider the problem of flow over a local obstacle — a step or a drop — at the bottom of a channel. The function $\eta(x)$ specifying the bottom relief possesses the following properties: $\eta(x) = 0$ for $|x| \ge d$; $\eta(x) \ne 0$ for $x \in (-d, d)$; the equality $\eta'(x) = 0$ holds at the unique point $x_0 \in (-d, d)$, where the function $\eta(x)$ has a local maximum or a local minimum. For definiteness, we assume that the fluid flow moves from left to right. The stationary flow in the region $x < -d$ to the left of the obstacle is given by the relations

$$u = \sqrt{2(F(\psi) - gh_l)}, \qquad H = 1/\sqrt{2(F(\psi) - gh_l)}\,,$$

where $F(\psi)$ is a specified positive function defined for $0 < \psi < \psi_1$; ψ_1 and h_l are specified positive constants. The free-surface level to the left of the obstacle h_l satisfies the equation

$$K(h_l) = h_l - \int\limits_0^{\psi_1} \frac{d\psi}{\sqrt{2(F(\psi) - gh_l)}} = 0.$$

3.1. *Supercritical and Transcritical Flows over a Local Obstacle at a Channel Bottom.* In this case, $F(\psi)$ and $h = h_l$ are specified at the entrance for $x = -d$. Thus, the inequalities $0 < h_l < h_c$ are satisfied. For the supercritical flow regime, where $K'(h) > 0$, the function $h(x)$ increases with increasing $\eta(x)$. Therefore, $K'(h)$ decreases with increasing $\eta(x)$ [since $K''(h) < 0$] but if $\eta(x) < K_{\max}$ everywhere, the sign of $K'(h)$ is retained and, after passing over the obstacle, the supercritical flow takes the initial parameter values. In this case, the flow parameters for $x = d$ and $x = -d$ coincide, in particular, the depth behind the obstacle $\delta = h$ coincides with the depth h_l ($0 < h_l < h_c$).

Let us consider the case where the value of the function $\eta(x)$ at the maximum point satisfies the equality $\eta(x_0) = K_{\max}$. As in the previous case, a subcritical solution is constructed in the region $x < x_0$. Because

$K'(h(x_0)) = 0$, for the continuation of the solution to the region $x > x_0$, there is an ambiguous choice of the supercritical or subcritical branches of the stationary solution. In the first version, just as in the previous case, the flow parameters take the initial values after passing over the obstacle. In the case of subcritical flow on the descending slope of the obstacle, the free-surface level $h(x)$ increases with decreasing $\eta(x)$, and if $K(h_0) < 0$, subcritical flow continues to $x = -d$, where the depth becomes equal to h_r. Here h_r is the unique solution of the equation $K(h) = 0$ on the subcritical branch $[K'(h_r) < 0,\ h_c < h_r]$.

If $K(h_0) > 0$, the solution $h = h(x)$ continues only to the point x_{00} ($x_0 < x_{00} < d$), where the equality $\eta(x_{00}) = K(h_0)$ holds. Next (in Sec. 4), it will be shown that the continuation of the stationary solution to the interval of $x \in (x_{00}, d)$ is possible, but in this case, regions of return flow appear on the descending slope of the obstacle.

For $\eta(x_0) > K_{\max}$, the problem of flow over an obstacle does not have a continuous solution.

In addition to the solutions listed above, discontinuous solutions are also possible. Let there be a discontinuous jump in the flow parameters at the point x_s ($-d < x_s < d$). We construct a stationary discontinuous solution for $x > x_s$ assuming that for $x < x_s$, a stationary supercritical solution has already been found (see Sec. 2). We first determine the limiting values of the discontinuous solution for $x \to x_s$ in the region behind the discontinuity front.

We use the following relations at the discontinuity jump

$$[u^2 - u_0^2] = 0, \qquad [uH] = 0, \qquad \Big[\int_0^{\psi_1} u^2 H\, d\psi + \frac{g\delta^2}{2}\Big] = 0; \tag{3.1}$$

these relations can be used to model small-amplitude discontinuities (see [4]). Here u_0 is the fluid velocity at the bottom (for $\psi = 0$), $\delta = h - \eta$ is the depth of the fluid layer, and $[f] = f^+ - f^-$ is the difference of the limiting values of the function f at the discontinuity front. We note that for flows without velocity shear, relations (3.1) imply the conditions at the discontinuities adopted in classical shallow-water theory. Denoting by the subscript 1 the limiting (for $x \to x_s$) values of the flow parameters in the region behind the jump, from the relations at the discontinuity $x = x_s$ we obtain

$$u_1^2(x_s, \psi) = u^2(x_s, \psi) - m, \qquad u_1 H_1 = 1,$$
$$\int_0^{\psi_1} u_1(x_s, \psi)\, d\psi + \frac{g\delta_1^2(x_s)}{2} = \int_0^{\psi_1} u(x_s, \psi)\, d\psi + \frac{g\delta^2(x_s)}{2}, \tag{3.2}$$

where $m = [u_0^2]$ is an unknown parameter that does not depend on ψ, $u(x_s, \psi)$ and $\delta(x_s)$ are known limiting (for $x \to x_s$) values of the functions $u(x, \psi)$ and $\delta(x)$ defined in the region $x < x_s$ ahead of the discontinuity. Using the relations on the discontinuity (3.1) and (3.2), we express the limiting values of the functions u_1, H_1, and δ_1 behind the discontinuity front in terms of their values ahead of the jump and the parameter m:

$$u_1(x_s, \psi) = \sqrt{u^2(x_s, \psi) - m}, \qquad H_1(x_s, \psi) = 1/\sqrt{u^2(x_s, \psi) - m}\,,$$
$$\delta_1(x_s) = \int_0^{\psi_1} \frac{d\psi}{\sqrt{u^2(x_s, \psi) - m}}. \tag{3.3}$$

The last relation in (3.2), which expresses the conservation law for the total momentum of the fluid layer, implies the equation for the quantity m

$$Q(m) - Q(0) = 0,$$

where

$$Q(m) = \int_0^{\psi_1} \sqrt{u^2(x_s, \psi) - m}\, d\psi + \frac{g}{2}\Big(\int_0^{\psi_1} \frac{d\psi}{\sqrt{u^2(x_s, \psi) - m}}\Big)^2. \tag{3.4}$$

We note that formulas (3.2)–(3.4) are meaningful only for those values of the parameter m that satisfy the inequality $m < m_0$, where $m_0 = \min_\psi u^2(x_s, \psi)$.

In seeking the flow parameters at the jump, it is necessary that in addition to relations (3.1), the stability condition for the discontinuous solution be also satisfied; i.e., it is necessary that the total energy of the fluid layer decrease when the flow passes through the discontinuity front (see [1, 4]). In [4], it is shown that this condition reduces to the inequality

$$l = u_1^2/2 + gh_1 - u^2/2 - gh < 0.$$

Using the representations (3.3) of the flow parameters at the discontinuity front, we determine the dependence of the quantity l on the parameter m:

$$l(m) = -\frac{m}{2} + g\Big(\int_0^{\psi_1} \frac{d\psi}{\sqrt{u^2(x_s,\psi) - m}} - \int_0^{\psi_1} \frac{d\psi}{u(x_s,\psi)}\Big).$$

The character of the increase or decrease of the functions $Q(m)$ and $l(m)$ is determined by the behavior of their derivatives:

$$Q'(m) = \frac{\delta_1}{2}\Big(g\int_0^{\psi_1} \frac{H_1\,d\psi}{u_1^2} - 1\Big), \qquad l'(m) = \frac{1}{2}\Big(g\int_0^{\psi_1} \frac{H_1\,d\psi}{u_1^2} - 1\Big). \tag{3.5}$$

We note that $Q'(m)$ necessarily changes sign as the parameter m varies from zero to m_0. Indeed, the quantity

$$S(m) = g\int_0^{\psi_1} \frac{H_1\,d\psi}{u_1^2} - 1 = g\int_0^{\psi_1} \frac{d\psi}{(u^2(x_s,\psi) - m)^{3/2}} - 1$$

for $m = 0$ is negative because the flow ahead of the jump is supercritical but $S(m)$ tends to $+\infty$ as $m \to m_0$. In view of the aforesaid and the inequality $S'(m) > 0$, it is concluded that there is a unique value of $m_c \in (0, m_0)$ such that $S(m_c) = 0$. In the interval $(0, m_c)$, the function $Q(m) - Q(0)$ decreases and reaches the negative minimum at the point m_c, and in the interval (m_c, m_0) it increases monotonically. If $Q(m_0) - Q(0) > 0$, there is a unique value $m_s \in (m_c, m_0)$ that satisfies the equation $Q(m_s) - Q(0) = 0$. Once m_s is found, all flow parameters behind the discontinuity front are calculated by the formulas given above, in which m needs to be replaced by m_s.

We show that for the solution found for $m_s > 0$, the energy decrease condition, namely, $l(m_s) < 0$, is satisfied. Indeed, integrating the corollary of relations (3.5) $l'(m) = Q'(m)/\delta_1$ from 0 to m_s, we obtain

$$l(m_s) = \int_0^{m_s} \frac{Q'(m)\,dm}{\delta_1} = \int_0^{m_s} \frac{(Q(m) - Q(0))\delta_1'\,dm}{\delta_1^2}. \tag{3.6}$$

This equality was derived using the relation $Q(m_s) - Q(0) = 0$. Taking into account that the inequalities $\delta_1'(m) > 0$ and $Q(m) - Q(0) < 0$ are satisfied at the intermediate points for $0 < m < m_s$, from (3.6) we obtain the inequality $l(m_s) < 0$ for $m_s > 0$. Using relation (3.6), it can be shown that the inequality $l(m_s) < 0$ holds only for $m_s > 0$ (see [4]).

To construct the flow in the region behind the discontinuity front at $x > x_s$, we assume that

$$F_1(\psi) = \frac{u_1^2(x_s,\psi)}{2} + gh_1(x_s) = \frac{u^2(x_s,\psi) - m_s}{2} + g\int_0^{\psi_1} \frac{d\psi}{\sqrt{u^2(x_s,\psi) - m_s}} + g\eta(x_s).$$

This equality can also be written as

$$F_1(\psi) = F(\psi) + l(m_s), \tag{3.7}$$

where

$$F(\psi) = \frac{u^2(x_s,\psi)}{2} + g\int_0^{\psi_1} \frac{d\psi}{u(x_s,\psi)} + g\eta(x_s).$$

Next, using the representation of the stationary solution (2.2), we express u and H in the region behind the jump in terms of the known function $F_1(\psi)$ and the unknown quantity h:

$$u = \sqrt{2(F_1(\psi) - gh)}, \qquad H = 1/\sqrt{2(F_1(\psi) - gh)}. \tag{3.8}$$

This yields the following equation for the function $h(x)$ in the region $x > x_s$:

$$K_1(h) = h - \int\limits_0^{\psi_1} \frac{d\psi}{\sqrt{2(F_1(\psi) - gh)}} = \eta(x). \tag{3.9}$$

In view of equality (3.7), $K_1(h)$ is transformed as

$$K_1(h) = h - \int\limits_0^{\psi_1} \frac{d\psi}{\sqrt{2(F(\psi) - g(h - l(m_s)/g))}},$$

and Eq. (3.9) is finally written as

$$K(h - l(m_s)/g) = \eta(x) - l(m_s)/g.$$

This formula implies that for a stationary solution to exist, it is necessary that the function $\eta(x)$ specifying the bottom relief should satisfy the inequality $\eta(x) \le K_{\max} + l(m_s)$ for $x > x_s$. In the case of the strict inequality, the flow over the top of the obstacle retains the nature of subcritical flow. If the inequality $-l(m_s)/g > K(h_0)$ is satisfied, the subcritical flow continues to the point $x = d$. If the inequality $-l(m_s)/g < K(h_0)$ is satisfied, the subcritical flow of the form (3.8) continues only to the point x_*, where the equality $K(h_0) = \eta(x_*) - l(m_s)/g$ holds. At the point $x = x_*$, the velocity u vanishes and the continuation of the stationary solution to the region $x > x_*$ includes a zone of recirculation flow on the opposite slope of the obstacle. In the case where the equality $\eta(x_0) - l(m_s) = K_{\max}$ holds at the point x_0, transition to the supercritical regime in the region $x \in (x_0, d)$ is possible.

The case $Q(m_0) - Q(0) < 0$ requires a separate consideration.

3.2. *Subcritical and Transcritical Flows over a Local Step at a Channel Bottom.* In this case, the free-surface level $h = h_l$ specified on the left boundary $x = -d$ satisfies the inequalities $h_c < h_l < h_0$. In the interval of $x \in (-d, d)$, the function $h(x)$ is defined using Eq. (2.3). According to the aforesaid in Sec. 2, the function $h(x)$ decreases with increasing $\eta(x)$. In this case, $K'(h)$ increases because $K''(h) < 0$. However, if the inequality $\eta(x) < K_{\max}$ is satisfied, the function $K'(h)$ does not change sign above the obstacle is executed and the flow is subcritical everywhere in the interval $(-d, d)$. The flow parameters on the right ($x = d$) and left ($x = -d$) boundaries of the obstacle coincide.

Let the function $\eta(x)$ be such that the equality $\eta(x_0) = K_{\max}$ holds at its maximum point, and the flow at $x < x_0$ is subcritical. Above the obstacle, the function $h(x)$ varies from h_l to h_c as x varies from $-d$ to x_0. For $x \to x_0$, the equality $K'(h(x_0)) = 0$ holds. The continuation of the solution for $x > x_0$ can be performed in two ways by choosing the subcritical or supercritical branches of the solution. Choosing the supercritical branch $h = h_2(x)$, we obtain transcritical flow above the obstacle. At the exit $x = d$, the depth becomes equal to h_r, where $h_r < h_l$ is a solution of the equation $K(h_r) = K(h_l)$.

In the case where $\eta(x_0) > K_{\max}$, a continuous stationary solution of Eq. (2.3) does not exist because $K(h) < K_{\max}$ for any values of h.

3.3. *Supercritical Flow over a Local Drop at a Channel Bottom.* Let us study the problem of flow over a drop at a channel bottom. If the incident flow is supercritical, then $K'(h) > 0$ and $h(x)$ decreases with decreasing $\eta(x)$. Then, $K'(h)$ increases and the flow remains supercritical up to the lowest point of the bottom. The stationary solution is defined by formulas (2.2). Behind the obstacle, the flow parameters take the initial values.

3.4. *Subcritical Flow over a Local Drop at a Channel Bottom.* We consider subcritical flow over a local drop local drop at a channel bottom. In this case, $K'(h) < 0$ in the flow incident on the obstacle. On the segment of decreasing $\eta(x)$, the function $h(x)$ increases; then, $K'(h)$ decreases. This implies that on the specified segment, the flow remains subcritical. If the function η satisfies the inequality $\eta(x_0) > K_0$, the subcritical flow continues to the boundary $x = -d$. After passing over the obstacle, the flow parameters take the initial values. If $\eta(x_0) < K_0$, the subcritical solution of the form, (2.2) continues only to the point $x_1 \in (-d, x_c)$. At the indicated point, $F(0) = gh$; therefore, the velocity at the bottom vanishes and the continuation of the solution of the form (2.2) becomes impossible because of the occurrence of negative quantities under the square root in formulas (2.2). Next, it will be shown that to continue the solution to the region $x > x_1$, one can use the stationary solutions describing recirculation-zone flows.

4. Recirculation-Zone Flows. In the case where $\eta(x) \to K_0$ for $x \to x_1$ and $\eta'(x) < 0$, the radicand in (2.2) vanishes and the continuation of the stationary solution for the subcritical flow to the region $x > x_1$ becomes impossible because the term under the square root is negative. Let us show that in this case, it is possible to construct stationary flow of different structure that includes recirculation zones, in which the fluid particles perform rotational motion. The solution is constructed using the methods proposed in [5].

For definiteness, we consider the case of subcritical flow over a local drop at the bottom where the function $\eta(x)$ vanishes at the ends of the interval $(-d, d)$ and reaches the value $\eta_{\min} < K_0 < 0$ at the point $x_0 \in (-d, d)$. Let the equalities $\eta(x_1) = \eta(x_2) = K_0$ hold at the points x_1, x_2 $(-d < x_1 < x_0 < x_2 < d)$. We show that there is a continuation of the stationary flow to the region $x > x_1$ such that there is a recirculation zone at $x_1 < x < x_2$ that occupies the region D defined by the inequalities $\eta(x) < y < \eta_1(x)$, $x_1 < x < x_2$, and in this case, $\eta_1(x_1) = \eta_1(x_2) = K_0$. In the region $y > \eta_1(x)$, the stationary solution is defined by the formulas

$$u = \sqrt{2(F(\psi) - gh)}, \qquad H = 1/\sqrt{2(F(\psi) - gh)}, \qquad K(h) = \eta_1(x), \tag{4.1}$$

where $K(h)$ is defined in (2.3). We assume that in the region D, the function u changes sign for $y = \eta_0(x)$; in this case, the horizontal velocity component is positive for $\eta_0(x) < y < \eta_1(x)$ and negative for $\eta(x) < y < \eta_0(x)$. In the region D, u is expressed in the variables x and ψ as

$$u = \pm\sqrt{2(G(\psi) - gh)}, \qquad H = \pm 1/\sqrt{2(G(\psi) - gh)} \tag{4.2}$$

with a certain function $G(\psi)$ defined for $\psi < 0$. We require that the velocity u be continuous for passage through the boundary $y = \eta_1(x)$. This condition is satisfied if the function $G(\psi)$ obeys the equality $G(0) = F(0)$.

Let $\psi = \psi_*(h)$ be a solution of the equation $G(\psi) - gh = 0$ $[\psi_*(h) < 0]$. The equality $\psi = \psi_*(h)$ is satisfied everywhere on the boundary $y = \eta_0(x)$ by the definition of this boundary. We obtain the relation linking the quantities h, η, η_0, η_1, and $G(\psi)$ in the recirculation zone. For this, we use the relations

$$\begin{gathered}\eta_0 - \eta = \int_0^{\psi_*(h)} H\, d\psi = -\int_0^{\psi_*(h)} \frac{d\psi}{\sqrt{2(G(\psi) - gh)}},\\ \eta_1 - \eta_0 = \int_{\psi_*(h)}^0 H d\psi = \int_{\psi_*(h)}^0 \frac{d\psi}{\sqrt{2(G(\psi) - gh)}}.\end{gathered} \tag{4.3}$$

Eliminating η_0 from relations (4.3) and using the equality $K(h) = \eta_1$, we obtain

$$\int_{\psi_*(h)}^0 \frac{d\psi}{\sqrt{2(G(\psi) - gh)}} = \frac{1}{2}\,(K(h) - \eta(x)).$$

Assuming that $G'(\psi) \neq 0$, we make the change of variables $s = G(\psi)/g$ in the last integral. As a result, we obtain

$$\int_h^A \frac{f(s)\, ds}{\sqrt{s - h}} = \frac{K(h) - \eta(x)}{\sqrt{2g}}. \tag{4.4}$$

Here $f(s) = 1/G'(\psi(s))$ and $A = g^{-1}F(0)$.

Let us show that the set of solutions of the problem of flow in the recirculation zone has an arbitrariness in one function of one variable. Indeed, we specify any monotonically increasing function $\tilde{\eta}(h)$ that is defined in $h \in (h_*, A)$ and takes the following values at the ends of the interval: $\tilde{\eta}(h_*) = \eta_{\min}$ and $\tilde{\eta}(A) = K_0$. Here $h_* = \mathrm{const}$ and $0 < h_* < A$. We next determine the function $h(x)$ in the interval (x_1, x_0) from the equation

$$\tilde{\eta}(h) = \eta_l(x), \tag{4.5}$$

where $\eta_l(x)$ is the restriction of the function $\eta(x)$ in the interval (x_1, x_0). Similarly, we determine $h(x)$ in the interval (x_0, x_2) as a solution of the equation

$$\tilde{\eta}(h) = \eta_r(x), \tag{4.6}$$

where $\eta_r(x)$ is the restriction of the function $\eta(x)$ in the interval (x_0, x_2).

Defining the function $h(x)$ for $x \in (x_1, x_2)$ and the function $\tilde{\eta}(h)$ in the interval (h_*, A), we find the function $G(\psi)$ in the recirculation-flow zone. Relation (1.4) can be written as

$$\int_h^A \frac{f(s)\,ds}{\sqrt{s-h}} = \varphi(h), \tag{4.7}$$

where

$$\varphi(h) = (K(h) - \tilde{\eta}(h))/\sqrt{2g}. \tag{4.7$'$}$$

The solution of the obtained Abel integral equation for the function $f(h)$ as the form

$$f(h) = \frac{1}{\pi}\Big(\frac{\varphi(A)}{\sqrt{A-h}} - \int_h^A \frac{\varphi'(s)\,ds}{\sqrt{s-h}}\Big).$$

In the examined case, $\varphi(A) = 0$ since the equality $K(A) = \tilde{\eta}(A) = \eta(x_1)$ is satisfied at $x = x_1$. Hence, the formula specifying the solution of Eq. (4.7) is simplified:

$$f(h) = -\frac{1}{\pi}\int_h^A \frac{\varphi'(s)\,ds}{\sqrt{s-h}}. \tag{4.8}$$

We note that the solution of the Abel equation obeys the relation

$$\int_h^A f(s)\,ds = \frac{1}{\pi}\int_h^A \frac{\varphi(s)\,ds}{\sqrt{s-h}}, \tag{4.9}$$

which will be used below.

To finish the construction of the solution in the recirculation zone, we find the function $G(\psi)$ as a solution of the Cauchy problem:

$$G'(\psi) = 1/f(g^{-1}G(\psi)), \qquad G(0) = gA. \tag{4.10}$$

Integrating the differential equation subject to the boundary condition, we obtain the equality

$$\chi(G) = \psi, \qquad \chi(G) = -g\int_{G/g}^A f(\xi)\,d\xi = -\frac{g}{\pi}\int_{G/g}^A \frac{\varphi(s)\,ds}{\sqrt{s-G/g}}, \tag{4.11}$$

which defines the function $G(\psi)$ implicitly. Relation (4.11) was derived using equality (4.9).

We note that the derivative $\chi'(G)$ does not vanish. Indeed, because $K'(h) < 0$ and $\tilde{\eta}'(h) > 0$ according to the choice of the function $\tilde{\eta}(h)$, the inequality $\varphi'(h) < 0$ is valid [see formula (4.7$'$)]. From formulas (4.7$'$) and (4.8) it follows that $f(h) > 0$; then, the functions $\chi'(G)$ and $G'(\psi)$ also take positive values. Using the monotonicity of the function $\chi(G)$, we uniquely determine the function $G(\psi)$ from the implicit relation (4.11). Thus, the solution with a recirculation zone is constructed. In this zone, the functions u and H are defined by formulas (4.2), and the function $h(x)$ is defined by relations (4.5) and (4.6). The solution obtained depends on the arbitrary function $\tilde{\eta}(h)$.

Remark 1. Instead of the function $\tilde{\eta}(h)$, in the interval (x_1, x_0) we can specify an arbitrary monotonically decreasing function $h = h(x)$ that satisfies the conditions $h(x_1) = A$ and $h(x_0) = h_*$. Inversion of the dependence $h = h(x)$ yields $x = X(h)$, where $x_1 < X(h) < x_0$. The function $\tilde{\eta}(h)$ is then defined in the interval (h_*, A) by the relation $\tilde{\eta}(h) = \eta(X(h))$, after which the function $h = h(x)$ is determined in the interval (x_0, x_2) as a solution of Eq. (4.6). The function $G(\psi)$ can also be specified for $\psi < 0$. In this case, the dependence $h = h(x)$ is found from Eq. (4.4).

Formulas for transforming to the plane of the variables x and y are given below. The boundary of the recirculation-flow region is given by the equation

$$y = \eta_1(x) = K(h(x)).$$

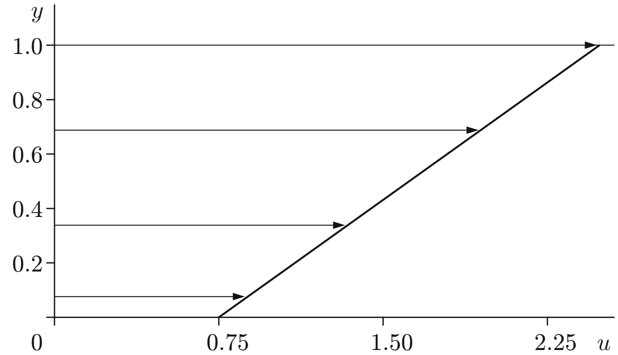

Fig. 2. Velocity profile over the depth.

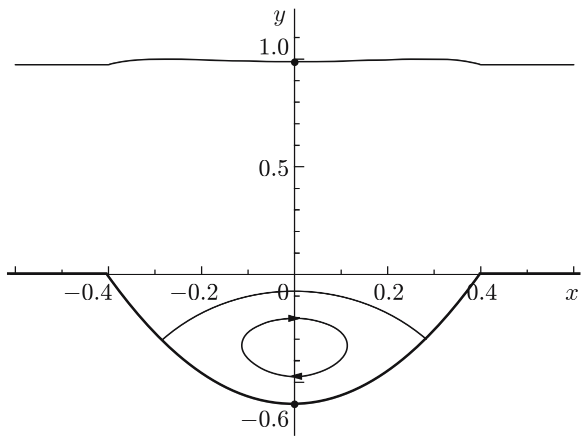

Fig. 3. Flow pattern.

The velocity u vanishes on the curve

$$y = (K(h(x)) + \eta(x))/2.$$

At $\eta(x) < y < \eta_0(x)$ in the region D, the streamlines $\psi = \psi_0 = \text{const}$ are given by the equations [ψ_0 varies in the interval $\psi_*(h_*) < \psi_0 < 0$]

$$y = \int_{\psi_0}^{0} \frac{d\psi}{\sqrt{2(G(\psi) - gh(x))}} + \eta(x) = \sqrt{\frac{g}{2}} \int_{G(\psi_0)/g}^{A} \frac{f(s)\,ds}{\sqrt{s - h(x)}} + \eta(x),$$

and in the subregion $\eta_0(x) < y < \eta_1(x)$ of the region D, they are given by the equations

$$y = K(h(x)) - \int_{\psi_0}^{0} \frac{d\psi}{\sqrt{2(G(\psi) - gh(x))}} = K(h(x)) - \sqrt{\frac{g}{2}} \int_{G(\psi_0)/g}^{A} \frac{f(s)\,ds}{\sqrt{s - h(x)}}.$$

We consider stationary shear flow of constant depth $h = 0.9739$ which is defined by relations (2.6) and passes over a local drop located at $-0.4 < x < 0.4$ at the bottom. The bottom topography at $x \in (-0.4, 0.4)$ is given by the equation

$$y = -0.6(1 - x^2/0.16).$$

The velocity profile for this flow is presented in Fig. 2.

Stationary flow is possible only in the case where return flow occurs at the local drop. Figure 3 gives calculation results for the free-surface shape, the boundary of the recirculation zone, and one of the closed streamlines inside this zone obtained under the assumption that the dependence of the Bernoulli constant on ψ in the recirculation zone is the same as that in the main flow $G(\psi) = 2.762\psi + 9.81$.

Remark 2. For the problem of flow over an obstacle at the bottom considered in Sec. 3.1, the construction of a solution with a recirculation zone is similar to that given above because the flow in the region of $x_{00} < x < d$ is defined in the same way as in half of the drop [regions $x \in (x_1, x_0)$], and at $x > d$, the flow is homogeneous shear flow with $h(x) = \text{const}$.

Thus, the concepts of flow subcriticality and supercriticality were extended to the case of stationary shear flows. The properties of the solutions of the long-wave equations were studied for different flow regimes. Exact solutions were obtained that describe wide classes of subcritical, supercritical, and transcritical shear flows above an uneven bottom.

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REFERENCES

1. V. Yu. Liapidevskii and V. M. Teshukov, *Mathematical Models for the Propagation of Long Waves in an Inhomogeneous Fluid* [in Russian], Izd. Sib. Otd. Ross. Akad. Nauk, Novosibirsk (2000).
2. V. M. Teshukov, "On the hyperbolicity of the long-wave equations," *Dokl. Akad. Nauk SSSR*, **284**, No. 3, 555–562 (1985).
3. V. E. Zakharov, "Benney's equations and quasiclassical approximation in the inverse problem method," *Funktsional. Anal. Prlozh.*, **14**, No. 2, 15–24 (1980).
4. V. M. Teshukov, "Hydraulic jump in the shear flow of an ideal incompressible fluid," *J. Appl Mech. Tech. Phys.*, **36**, No. 1, 10–18 (1995).
5. E. Varley and P. A. Blythe, "Long eddies on sheared flows," *Stud. Appl. Math.*, **68**, 103–187 (1983).